SEVERAL PROPERTIES OF THE LAPLACE TRANSFORM AND STRUCTURE OF SOLUTIONS OF BOUNDARY-VALUE PROBLEMS OF THERMAL CONDUCTIVITY WITH MEMORY

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Several properties of the Laplace transform, used in constructing solutions of boundary-value problems of thermal conductivity on the basis of equations with memory, are established.

The hyperbolic transport equation and equations with memory are currently used in treating short-lived high-intensity processes of heat and mass transfer [1-4]. Since a large number of solutions of boundary-value problems has been accumulated for various geometries, it is advisable to use them in solving boundary-value problems of thermal conductivity on the basis of more complicated transport laws. With this purpose we derive below several properties of the Laplace transform, by means of which relationships are established between solutions of boundary-value problems of thermal conductivity using various transport laws.

Several Properties of the Laplace Transform. For an arbitrary function $u_1(t)$ we introduce the direct L_φ and inverse $L_{\varphi(t)}^{-1}$ Laplace transforms in the function $\varphi(p)$:

$$u_1(t) = L_{\varphi(t)}^{-1} U_1(\varphi) = \frac{1}{2\pi i} \int\limits_{\sigma - i\infty}^{\sigma + i\infty} \exp[\varphi(p)t]\, U_1[\varphi(p)]\, d\varphi(p),$$

$$U_1[\varphi(p)] = L_\varphi u_1(t) = \int\limits_0^\infty \exp[-\varphi(p)t]\, u_1(t)\, dt. \tag{1}$$

The additional index (t) at the operator L_φ^{-1} of the inverse transform indicates that it is taken over the argument t, enclosed in the circular brackets. The function $u_1(t)$ must satisfy requirements imposed on the inverse transform, and the analytic function $\varphi(p)$ — the imaging condition, i.e., $\mathrm{Re}[\varphi(p)] > \sigma$ for $\mathrm{Re}(p) > \sigma$, where σ is the growth index of the function $u_1(t)$. Transform (1) can be considered as a formal replacement of p by φ in the ordinary Laplace transforms L_p and L_p^{-1}; therefore, all properties of the ordinary Laplace transforms are valid for L_φ and L_φ^{-1}.

If in the Laplace transform $U(p) = L_p u$ of the function $u(t)$ one replaces the variable p by an arbitrary function $\varphi = \varphi(p)$ by means of the identity

$$U(p) = U[p(\varphi)] = U_1[\varphi(p)],$$

the originals $u(t)$ and $u_1(t)$ of these transforms are related by

$$u(t) = \int\limits_0^\infty u_1(\tau)\, a(t, \tau)\, d\tau, \tag{2}$$

where

$$a(t, \tau) = L_{p(t)}^{-1} \{\exp[-\varphi(p)\tau]\}. \tag{2a}$$

Here the analytic function $\varphi(p)$ satisfies the imaging condition, and the integral is assumed to be uniformly convergent for $t \in [0, \infty)$. Relationship (2) is easily proved, applying

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to it the Laplace transform L_p, changing the order of integration, and taking into account Eq. (2a):

$$U(p)=L_{p(t)}\int\limits_0^{\infty} u_1(\tau)\,a(t,\ \tau)\,d\tau=\int\limits_0^{\infty} u_1(\tau)\,d\tau\int\limits_0^{\infty}\exp(-pt)\,a(t,\ \tau)\,dt=\int\limits_0^{\infty}\exp[-\varphi(p)\tau]\,u_1(\tau)\,d\tau=L_{\varphi}u_1=U_1[\varphi(p)].$$

The function $u_1(t)$, in turn, can be expressed in terms of $u(t)$ by means of the equation

$$u_1(t)=\int\limits_0^{\infty} u(\tau)\,a_0(t,\ \tau)\,d\tau,\qquad a_0(t;\ \tau)=L^{-1}_{\varphi(t)}\{\exp[-p(\varphi)\tau]\}. \tag{3}$$

We write down the generalized rule of Laplace transform multiplication. Let two transforms $U_1(p)$ and $G_1(p)$ and two analytic functions $\varphi(p)$ and $\eta(p)$ be given, satisfying the mapping condition. The following relation is then valid:

$$L_p^{-1}\{U_1[\varphi(p)]\,G_1[\eta(p)]\}=\int\limits_0^t d\tau\int\limits_0^{\infty} u_1(\xi)\,a(\tau,\ \xi)\,d\xi\int\limits_0^{\infty} g_1(\theta)\,b_1(t-\tau,\ \theta)\,d\theta, \tag{4}$$

where

$$a(\tau,\ \xi)=L^{-1}_{p(\tau)}\{\exp[-\varphi(p)\xi]\};\qquad b_1(t-\tau,\ \theta)=L^{-1}_{p(t-\tau)}\{\exp[-\eta(p)\theta]\}. \tag{4a}$$

Relations (4), (4a) are proved by applying the Borel theorem and Eqs. (2), (2a) under the same assumptions. Putting $\eta(p) = p$, relations (4), (4a) transform to the Efros transform, written in a somewhat different form:

$$L_p^{-1}\{U_1[\varphi(p)]\,G_1(p)\}=\int\limits_0^{\infty} u_1(\xi)\Big[\int\limits_0^t g_1(t-\tau)\,a(\tau;\ \xi)\,d\tau\Big]d\xi.$$

Applying mathematical induction, the Borel convolution theorem, and Eqs. (2), (4), one can write down an equation for the generalized product of several transforms. Using the Laplace transform properties and Eq. (2), one can derive other properties, useful for practical search of originals. Thus, applying Eq. (2) and the convolution theorem, we write down an expression for the original of the following transform:

$$u(t)=L_p^{-1}U_1[\varphi_1(p)+\varphi_2(p)]=\int\limits_0^{\infty} u_1(\tau)\,a(t,\ \tau)\,d\tau,\qquad a(t,\ \tau)=\int\limits_0^t a_1(\xi,\ \tau)\,a_2(t-\xi,\ \tau)\,d\xi,\qquad a_i(\xi,\ \tau)=L^{-1}_{p(\xi)}\{\exp[-\varphi_i(p)\tau]\},\ i=1,\ 2, \tag{5}$$

which we will need in studying the behavior of originals for short times. The equations obtained are useful both for practical search of originals, and for constructing solutions of thermal conductivity problems with memory.

Relation between Thermal Conductivity Problems, Based on Various Transport Laws. Problems of thermal conductivity have been considered lately, based not only on the linear Fourier transport law:

$$q=-\frac{\lambda_0}{l}\nabla T, \tag{6}$$

but also on the linearized relations describing transport processes with thermal memory [1, 2]. We write them, as well as (6), in dimensionless form:

$$q=-\frac{\lambda_0}{l}\lambda_1(0)\nabla T-\frac{\lambda_0}{l}\int\limits_0^\infty \frac{d\lambda_1(\mathrm{Fo}')}{d\,\mathrm{Fo}'}\nabla T(\mathrm{Fo}-\mathrm{Fo}',\ M)\,d\,\mathrm{Fo}',$$

$$e=e_0+\rho_0 c_0\left\{c_1(0)\,T+\int\limits_0^\infty \frac{dc_1(\mathrm{Fo}')}{d\,\mathrm{Fo}'}\ T(\mathrm{Fo}-\mathrm{Fo}',\ M)\,d\,\mathrm{Fo}'\right\}. \tag{6a}$$

Here $\lambda_1(\mathrm{Fo})$ and $c_1(\mathrm{Fo})$ are the dimensionless relaxation functions of thermal flow and internal energy. The case of the hyperbolic equation corresponds to a specific form of the relaxation function [2]. From Eqs. (6) and the internal energy conservation law follow the equations of thermal conductivity for the function $u(\mathrm{Fo},\ M) = T(\mathrm{Fo},\ M) - T_0$, where $T_0 = T(0,\ M)$, corresponding to a different transport laws:

$$-\frac{\partial u}{\partial\,\mathrm{Fo}}+\Delta u=-\frac{l^2}{\lambda_0}\,b(\mathrm{Fo},\ M), \tag{7}$$

$$-c_1(0)\frac{\partial u}{\partial\,\mathrm{Fo}'}-\int\limits_0^\infty \frac{dc_1(\mathrm{Fo}')}{d\,\mathrm{Fo}'}\,\frac{\partial u(\mathrm{Fo}-\mathrm{Fo}',\ M)}{\partial\,\mathrm{Fo}}\,d\,\mathrm{Fo}'+\lambda_1(0)\,\Delta u+\int\limits_0^\infty \frac{d\lambda_1(\mathrm{Fo}')}{d\,\mathrm{Fo}'}\,\Delta u(\mathrm{Fo}-\mathrm{Fo}',\ M)\,d\,\mathrm{Fo}'=-\frac{l^2}{\lambda_0}\,b(\mathrm{Fo},\ M), \tag{7a}$$

where $u(0,\ M) = 0$ and M is a point of space. If the boundary-value problem of thermal conductivity is solved for the region D with boundary Γ, one of the following two boundary conditions must be assigned:

$$u(\mathrm{Fo},\ M)=u_0(\mathrm{Fo},\ M),\ M\in\Gamma, \tag{8}$$

$$q(\mathrm{Fo},\ M)=q_0(\mathrm{Fo},\ M),\ M\in\Gamma. \tag{9}$$

Here u_0 and q_0 are the temperature of the medium and the external thermal flow.

Applying the Laplace transform to Eq. (7), we write in the mapping region

$$\Delta U(p,\ M)-\varphi(p)\,U(p,\ M)=-\frac{l^2}{\lambda_0}B(p,\ M),\ M\in D, \tag{10}$$

where

$$\varphi(p)=p;\ B(p,\ M)=L_p b(\mathrm{Fo},\ M); \tag{10a}$$

$$\varphi(p)=pC_1(p)/\Lambda_1(p);\ B(p,\ M)=\frac{L_p b(\mathrm{Fo},\ M)}{p\Lambda_1(p)}. \tag{10b}$$

Transforming in Eqs. (10), (10b) from the variable p to the function $\varphi(p)$ in the source transforms, and solving by means of the identity $B(p,\ M) = B_1[\varphi(p),\ M]$, $U(p,\ M) = U_1[\varphi(p),\ M]$, for the mapping $U_1(\varphi,\ M)$ Eq. (10) acquires the same form as in the case of the Fourier transport law, but with a source $B_1(\varphi,\ M)$:

$$\Delta U_1(\varphi,\ M)-\varphi U_1(\varphi,\ M)=-\frac{l^2}{\lambda_0}\ B_1(\varphi,\ M),\quad M\in D. \tag{11}$$

The solution of thermal conductivity problems without boundary conditions (such as finding potentials, solutions for the exact instantaneous source, etc.) is completely determined by Eq. (11). Applying Eqs. (2), (2a) to the mapping $U_1[\varphi(p),\ M]$, we obtain a solution of the original problem of thermal conductivity for Eqs. (7a) in form of the integral

$$u(\mathrm{Fo},\ M)=\int\limits_0^\infty u_1(\mathrm{Fo}',\ M)\,a(\mathrm{Fo},\ \mathrm{Fo}')\,d\,\mathrm{Fo}',$$

$$a(\mathrm{Fo},\ \mathrm{Fo}') = L^{-1}_{p(\mathrm{Fo})}\{\exp[-\varphi(p)\,\mathrm{Fo}']\}. \tag{12}$$

Thus, the solution u(Fo, M) for the equation of thermal conductivity with a memory is determined by the solution u_1 of a problem of the same type for the ordinary equation of thermal conductivity (7) with a respective change of the source and the shape of relaxation functions, on which the second factor in the integral depends. To obtain the function $U_1(\varphi, M)$ in the case of a homogeneous Eq. (7) it is sufficient to formally replace p by φ in the transform solution for the usual equation of thermal conductivity, i.e., the function $u_1(\mathrm{Fo}, M)$ is the solution of the same problem for the parabolic equation of thermal conductivity. Potentials were derived by the same principle for the hyperbolic equation of thermal conductivity [5]. In the case of an equation with a source one can write by means of Eq. (3) an expression, relating the equivalent source $b_0(\mathrm{Fo}, M)$ of the parabolic equation of thermal conductivity with the source b(Fo, M) of Eq. (7a) with thermal memory

$$b_0(\mathrm{Fo},\ M) = \int_0^\infty a_0(\mathrm{Fo},\ \mathrm{Fo}') \int_0^{\mathrm{Fo}'} b(\mathrm{Fo}'',\ M)\, m_1(\mathrm{Fo}'-\mathrm{Fo}'')\, d\mathrm{Fo}''\, d\mathrm{Fo}',$$

$$a_0(\mathrm{Fo},\ \mathrm{Fo}') = L^{-1}_{\varphi(\mathrm{Fo})}\{\exp[-p(\varphi)\,\mathrm{Fo}']\}, \tag{13}$$

$$m_1(\mathrm{Fo}-\mathrm{Fo}') = L^{-1}_{p(\mathrm{Fo}-\mathrm{Fo}')}\left\{\frac{1}{p\Lambda_1(p)}\right\}.$$

The boundary condition (8) has in the mapping region an identical form for (8a), (8b):

$$U(p,\ M) = U_0(p,\ M),\quad M\in\Gamma, \tag{14}$$

and the boundary condition (9) of the second kind can be written as follows, taking into account Eq. (6):

$$\frac{\lambda_0}{l}\,\frac{\partial U(p,\ M)}{\partial n} = s(p)\,Q_0(p,\ M),\quad M\in\Gamma, \tag{15}$$

where according to the transport law

$$s(p) = 1, \tag{15a}$$

$$s(p) = 1/p\Lambda_1(p), \tag{15b}$$

and n is the internal normal to the boundary Γ. The solution U(p, M) of the boundary-value problem is determined in the mapping region for the Fourier transport law by Eqs. (10), (10a) and boundary condition (14) [or (15), (15a)]. Transforming in boundary conditions (14) [or (15), (15b)] from the variable p to the function φ (p)

$$U_1[\varphi(p),\ M] = U_0(p,\ M) = U_{01}[\varphi(p),\ M],\quad M\in\Gamma, \tag{16}$$

$$\frac{\lambda_0}{l}\,\frac{\partial U_1[\varphi(p),\ M]}{\partial n} = Q_{01}[\varphi(p),\ M],\quad M\in\Gamma, \tag{17}$$

where

$$Q_{01}[\varphi(p),\ M] = s[p(\varphi)]\,Q_0[p(\varphi),\ M], \tag{17a}$$

then Eq. (11) for mapping U_1 together with boundary condition (16) [or (17)] for the case of heat transport with memory acquires the same form as for the Fourier transport law. The solution of the boundary-value problem for Eq. (7a) is primarily determined by Eq. (12), in which the function $u_1(\mathrm{Fo}, M)$ is a solution of a boundary-value problem of the same type for the ordinary equation of thermal conductivity (7) with a source b_0 and with one of the corresponding boundary conditions:

$$u_1(\mathrm{Fo},\ M) = \int_0^\infty u_0(\mathrm{Fo}',\ M)\, a_0(\mathrm{Fo},\ \mathrm{Fo}')\, d\mathrm{Fo}',\quad M\in\Gamma, \tag{18}$$

$$\frac{\lambda_0}{l}\frac{\partial u_1(\mathrm{Fo},\ M)}{\partial n}=\int_0^\infty a_0(\mathrm{Fo},\ \mathrm{Fo}')\int_0^{\mathrm{Fo}'} q_0(\mathrm{Fo}'',\ M)\,m_1(\mathrm{Fo}'-\mathrm{Fo}'')\,d\,\mathrm{Fo}'d\,\mathrm{Fo}''. \tag{19}$$

Since a large number of problem solutions has been accumulated for the parabolic equation for various regions D, the representation (12) can facilitate analysis and solution of boundary-value problems for equations with thermal memory. We note that replacing p by φ (p) in the mapping U(p, M) of the boundary-value problem for the homogeneous parabolic equation of thermal conductivity (7) is equivalent to including a thermal memory in Eq. (7a) and neglecting the effect of thermal memory in the boundary conditions.

Equations (10)-(19) establish the reciprocal relation between thermal conductivity problems with and without thermal memory. This analogy can be used as a correspondence of the original solutions, i.e., the use of Eqs. (12), (13), (18), and (19) for the very solution of problems of thermal conductivity, heat sources, and boundary conditions. This method is suitable for the functions $\varphi(p)$, satisfying the mapping condition, and is effective for practical applications only for regions D having a convenient solution of the parabolic equation of thermal conductivity for arbitrary functions appearing in the boundary condition (and for an arbitrary form of the source, depending on coordinates and on time in solving problems with a source). More effective is the use of the correspondence between the transforms U(p, M) and $U_1(\varphi, M)$ of the boundary-value problem for the ordinary equation of thermal conductivity and the equation with memory. In this case it is necessary to replace p in the Laplace transforms of the boundary conditions U(p, M) and $\partial U(p, M)/\partial n$ and in the source B(p, M) [see Eqs. (14), (15), and (10)] by the inverse function $p = p(\varphi)$ [Eqs. (16), (17)], and replace p by φ in the equation itself. As a result of this replacement the problem in the mapping L_φ [by the function $\varphi(p)$] acquires the same form as for the ordinary equation of thermal conductivity. Solving the problem obtained by the ordinary operator method and using the operator L_φ^{-1}, we find a solution $u_1(\mathrm{Fo}, M)$; later we determine from Eq. (12) the solution of the boundary-value problem for the original equation with thermal memory. In this case the correspondence between the mapping can be used more flexibly and fully with available information on solutions of problems of similar type for the parabolic equation of thermal conductivity. It is important to note that this method of solving the problem makes it possible to use a function $\varphi(p)$ not satisfying the mapping condition.

Thus, in using the hyperbolic heat equation in a boundary-value problem of the first kind for a semiinfinite bar

$$\begin{gathered}\eta\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-a\frac{\partial^2 u}{\partial x^2}=0,\ \eta=a\beta^2,\quad x\in[0,\ \infty),\\ u(0,\ x)=\frac{\partial u(0,\ x)}{\partial t}=0,\ u(t,\ \infty)=0,\ u(t,\ 0)=u_0(t)\end{gathered} \tag{20}$$

the function $\varphi(p)$, being a complete second-order expression $\varphi(p) = \eta p^2 + p$, does not satisfy the mapping condition. Therefore, one cannot return from the mapping problem $U_1(\varphi, x) = U_0[p(\varphi)]\exp[-x\sqrt{\varphi(p)}/\sqrt{a}]$ to the region of originals $L_p^{-1}U_1(\varphi, x)$ by Eq. (12). Transforming to the function $\varphi_0(p) = \sqrt{\varphi(p)} = \sqrt{\eta p^2 + p}$, it satisfies the mapping condition; therefore, from the mapping solution

$$\begin{gathered}U_1[\varphi_0(p),\ x]=U_{01}(\varphi_0)\exp[-x\varphi_0(p)/\sqrt{a}],\\ U_{01}(\varphi_0)=U_0[p(\varphi_0^2)],\ p(\varphi_0^2)=-\frac{1}{2\eta}+\sqrt{\frac{\varphi_0^2}{\eta}+\frac{1}{4\eta^2}}\end{gathered} \tag{21}$$

one can transform to the region of originals $L_p^{-1}U_1(\varphi_0, x)$. Since in the given case

$$\begin{gathered}a(t,\ \tau)=L_{p(t)}^{-1}\{\exp[-\varphi_0(p)\,\tau]\}=\exp(-\tau/2\sqrt{\eta})\,\delta(t-\sqrt{\eta}\,\tau)+\\ +\frac{\tau}{2\sqrt{\eta}}\exp(-t/2\eta)\frac{\mathrm{I}_1\left(\frac{1}{2\eta}\sqrt{t^2-\eta\tau^2}\right)}{\sqrt{t^2-\eta\tau^2}},\end{gathered} \tag{22}$$

Eq. (12) acquires the following form:

$$u(t, x) = \exp(-t/2\eta)\left\{u_1(t/\sqrt{\eta}, x)/\sqrt{\eta} + \int_0^{t/\sqrt{\eta}} \frac{\tau}{2\sqrt{\eta}} \frac{I_1\left(\frac{1}{2\eta}\sqrt{t^2-\tau^2\eta}\right)}{\sqrt{t^2-\tau^2\eta}} u_1(\tau, x)\, d\tau\right\},$$

$$u_1(t, x) = L_{\varphi_0}^{-1} U_1(\varphi_0, x). \tag{23}$$

Equation (23) differs from the general form of the solution obtained in [6], but the results of solving specific problems by different methods are identical. For example, for a thermal shock, when $u_0(t) = \delta(t)$, Eq. (23) is

$$u(t, x) = \exp(-x/2a\beta)\,\delta(t-\beta x) + \frac{\beta x}{2\eta}\exp(-t/2\eta)\frac{I_1\left(\frac{1}{2\eta}\sqrt{t^2-\beta^2x^2}\right)}{\sqrt{t^2-\beta^2x^2}}.$$

This result coincides with the solution of the same problem by the equation of [6]. Many applications of the method developed here for various relaxation kernels will be given in later publications. The equations obtained can be applied to take into account the effect of thermal memory in various problems relating heat and mass exchange with thermoelasticity, in which the effect of mass exchange and mechanical characteristics on the temperature field are neglected.

NOTATION

λ_0, thermal conductivity; ρ_0, density of the material; c_0, heat capacity of the material, l, characteristic size of the region; M, point of the region; Fo, Fourier number; p, Laplace variable; $\lambda(\mathrm{Fo})$, $c(\mathrm{Fo})$, relaxation functions of the thermal flux and of the internal energy; $c_1(\mathrm{Fo}) = c(\mathrm{Fo})/c_0\rho_0$, relative relaxation function of internal energy; $\lambda_1(\mathrm{Fo}) = \lambda(\mathrm{Fo})/\lambda_0$, relative relaxation function of the thermal flux; T, temperature; and β, reciprocal of the heat propagation velocity.

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